An introduction to the Batalin-Vilkovisky formalism

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Abstract

The aim of these notes is to introduce the quantum master equation $\{S,S\}-2i\hbar\Delta S=0$, and to show its relations to the theory of Lie algebras representations and to perturbative expansions of Gaussian integrals. The relations of the classical master equation $\{S,S\}=0$ with the BRST formalisms are also described. Being an introduction, only finite-dimensional examples will be considered.

1 Introduction

The Batalin-Vilkovisky formalism is an algebraic/geometric setting to deal with asymptotic expansions of Gaussian integrals of the form

$$\int_{V} \Psi e^{\frac{i}{\hbar}S} dv$$

in presence of a Lie algebra \mathfrak{g} of infinitesimal symmetries of the action S. From a geometric point of view, the Batalin-Vilkovisky formalism is the theory of smooth functions on odd symplectic supermanifolds; from an algebraic point of view is a way of looking to Lie algebra representations (up to homotopy).

These notes are a written version of the lectures the author gave during the seventh edition of the "Rencontres Mathématiques de Glanon", held in Glanon (France) in Summer 2003. They are by no means complete. Rather, they have to be intended as an invitation to the classical papers [BaVi], [Schw] and [AKSZ].

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2 A familiar example

The idea at the heart of the Batalin-Vilkovisky formalism is essentially contained in the following example, which everyone is familiar with: compute

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 1} \mathrm{d}x$$

by the residues theorem. If we look closely to this computation, a lot of interesting features appear. To begin with, to compute a 1-real variable integral, we double the number of variables. Moreover, going from $\mathbb R$ to $\mathbb C\simeq\mathbb R^2$ something highly non-trivial has happened: the 1-form $\mathrm{d}z/(z^2+1)$ extends in a non-trivial way the 1-form $\mathrm{d}x/(x^2+1)$ to a closed 1-form on $\mathbb R^2$. From a real variable point of view we have changed an integral of the form

$$\int_{-\infty}^{+\infty} \varphi(x) \mathrm{d}x$$

to one of the form

$$\int_{\mathbb{R}\times\{0\}}\omega$$

where ω is a 1-form on \mathbb{R}^2 such that

$$\begin{cases} \omega \big|_{\mathbb{R} \times \{0\}} = \varphi(x) dx \\ d\omega = 0 \end{cases}$$

The domain of $\omega = \mathrm{d}z/(z^2+1)$ is $\mathbb{C}\setminus\{\pm i\}$. Moreover ω extends to ∞ , so it is actually a 1-form on $\mathbb{P}^1(\mathbb{C})\setminus\{\pm i\}$. The integration domain $\mathbb{R}\cup\{\infty\}$ is a cycle in $\mathbb{P}^1(\mathbb{C})\setminus\{\pm i\}$. Since ω is closed, we can compute our integral by choosing another cycle in the same homology class as $\mathbb{R}\cup\{\infty\}$. If we denote by Γ a little circle around i, then

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 1} \mathrm{d}x = \int_{\Gamma} \omega.$$

Now, in a little neighborhood of the point i, we can expand ω into its Laurent series:

$$\frac{\mathrm{d}z}{z^2 + 1} = \left(-\sum_{n=-1}^{\infty} \left(\frac{i}{2}\right)^{n+2} \zeta^n\right) \mathrm{d}\zeta; \qquad \zeta = z - i$$

So, if we denote by Γ_0 the circle $\Gamma - i$, we find

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 1} dx = -\sum_{n = -1}^{\infty} \left(\frac{i}{2}\right)^{n+2} \int_{\Gamma_0} \zeta^n d\zeta = \frac{1}{2} \int_0^{2\pi} dt = \pi$$

Let us analyze the various steps in the computations above.

- 1) The problem. We have to compute an integral $\int_M \Phi$, where M is an n-dimensional differential manifold, and Φ is a top dimensional form on M.
- 2) Doubling the coordinates. We embed M as a cycle into a 2n-dimensional manifold N and extend Φ to a closed n-form Ω on N. The condition

 $d\Omega = 0$ is a gauge invariance condition; the manifold M inside N is a gauge fixing.

- 3) Varying the cycle. We choose another cycle M_0 in N, in the same homology class as M. Since Ω is closed we can compute the original integral by integrating Ω over M_0 . Changing the integration cycle from M to M_0 is a change of gauge. The fact that the integral does not change is precisely the independence of the result from the gauge chosen.
- 4) The perturbative expansion. The cycle M_0 is chosen is such a way that the *n*-form Ω has a power series expansion in a neighborhood of M_0 . By exchanging series and integral and obtain a series expansion for the original integral.

In the example above, the closed 1-form ω has been found by a formal change of coordinates $x\mapsto z=x+i\,y$. In the BV formalism, this way of producing closed forms is called the *superfield* formalism.

3 Super vector spaces

A super-vector space (superspace for short) is a $(\mathbb{Z}/2\mathbb{Z})$ -graded vector space $V = V_0 \oplus V_1$. Here, V_0 denotes the vector space of *even* vectors and V_1 the vector space of *odd* vectors. The $(\mathbb{Z}/2\mathbb{Z})$ -grading is called *degree* or *parity*; the degree of a homogeneous element x will be denoted by the symbol \overline{x} .

Equivalently, one can think of a superspace as an ordinary vector space V endowed with an automorphism $\alpha \colon V \to V$ such that $\alpha^2 = \mathrm{id}$. In this case, V_0 is the 1-eigenspace and V_1 is the (-1)-eigenspace. Note that $\alpha(x) = (-1)^{\overline{x}}x$ for a homogeneous element.

The dimension of a super vector space is the pair $(\dim V_0, \dim V_1)$. If $\dim V_i = m_i$, one says that V is a $m_0|m_1$ -dimensional super vector space. All vector spaces considered in this note will be finite-dimensional.

Any vector space V can be considered a super vector space by taking $\alpha = \mathrm{id}$. One usually refers to this as "placing V in even degree and writes $V \oplus 0$ or simply V to denote the superspace (V,id) . Similarly, one can place V in odd degree; one writes $0 \oplus V[1]$ or simply V[1] to denote the superspace $(V,-\mathrm{id})$.

Another classical example of superspace is the following. Given a \mathbb{Z} -graded vector space $V=\bigoplus_{n\in\mathbb{Z}}V_n$, one can look at V as the superspace with

$$V_{\mathbf{0}} = \bigoplus_{n \in 2\mathbb{Z}} V_n; \qquad V_{\mathbf{1}} = \bigoplus_{n \in 2\mathbb{Z}+1} V_n$$

For instance, if (V_*, ∂) is a complex with a degree 1 differential, then ∂ can be seen as

$$\partial: V_0 \to V_1$$

 $\partial: V_1 \to V_0$

Now that we have defined the objects, we have to define morphisms, in order to make superspaces a category. As one could imagine, a morphism $\varphi \colon V \to W$ between two superspaces is a linear map preserving the

grading. Equivalently, φ is a linear map which intertwines α_V and α_W , i.e, such that the diagram

$$\begin{array}{c} V \xrightarrow{\alpha_V} V \\ \varphi \downarrow & & \downarrow \varphi \\ W \xrightarrow{\alpha_W} W \end{array}$$

is commutative.

The category of super vector spaces (on a fixed field \mathbb{K} , which in these notes will always be \mathbb{R} or \mathbb{C}) will be denoted by the symbol SuperVect. Changing the parity of vectors in a superspace defines an endofunctor on Supervect which will be denoted by Π :

$$\Pi: (V, \alpha) \mapsto (V, -\alpha)$$

that is,

$$(\Pi V)_0 = V_1; \qquad (\Pi V)_1 = V_0.$$

We have already seen that, if V is a vector space, then V can be seen as a super vector space concentrated in even degree. The super vector space V[1] obtained from V concentrating it in odd degree can be seen as ΠV .

The category $\mathsf{SuperVect}$ has a natural symmetric tensor category structure defined by

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$$
$$(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$$

Equivalently,

$$\alpha_{V\otimes W} = \alpha_V \otimes \alpha_W$$

The *braiding* operator

$$\sigma_{VW} \colon V \otimes W \to W \otimes V$$

is defined as

$$\sigma \colon x \otimes y \mapsto (-1)^{\overline{x} \cdot \overline{y}} (y \otimes x)$$

on homogeneous vectors x, y and then extended by linearity. Note that there is a natural embedding of symmetric tensor categories

$$\mathsf{Vect} o \mathsf{Supervect}$$
 $V \mapsto (V, \mathrm{id})$

The symmetric tensor category structure on Supervect implies that, for any super vector space V and any positive integer n there is a natural action of the symmetric group S_n on the nth tensor power of V. The nth symmetric power of V is defined as

$$S^{n}(V) = (V^{\otimes n})_{S_{n}} = V^{\otimes n}/\{x_{1} \otimes \cdots \otimes x_{n} - \sigma(x_{1} \otimes \cdots \otimes x_{n}), \sigma \in S_{n}\}$$

Note that

$$S^n(V_0 \oplus 0) = S^n(V_0)$$

and

$$S^n(0 \oplus V_1) = \bigwedge^n V_1,$$

so that symmetric powers of superspaces are a unified language for symmetric and exterior powers of ordinary vector spaces. For a general super vector space $V = V_0 \oplus V_1$ one has

$$S^{n}(V_{0} \oplus V_{1}) = \bigoplus_{k=0}^{n} \left(S^{k}(V_{0}) \otimes \bigwedge^{n-k} V_{1} \right)$$

4 The space of functions on a superspace

In what follows we will be mostly concerned with the space of regular functions on a superspace V. Since we are not interested in topological questions here, we will define regular functions as formal power series.

As a preliminary remark, note that the dual of a superspace is a superspace, via

$$\alpha_{V^*} = (\alpha_V)^*$$

This implies that the even linear functionals on V are those functionals that are zero on odd vectors, and odd linear functionals on V are those functionals that are zero on even vectors.

The space of regular functions on V is

$$\mathcal{F}(V) = \varinjlim_n S^n(V^*)$$

The definition of $\mathcal{F}(V)$ can be made more explicit by the use of supercoordinates. Let V be a $m_0|m_1$ dimensional super vector space, and let $\{e_1,\ldots,e_{m_0}\}$ be a basis of V_0 and $\{\varepsilon_1,\ldots,\varepsilon_{m_1}\}$ be a basis of V_1 . Then the dual basis $\{x^1,\ldots,x^{m_0},\theta^1,\ldots,\theta^{m_1}\}$ is a basis of V^* , with x^i even and θ^k odd. The linear functionals x^i are called *even coordinates* on Vand the functionals θ^k are called *odd coordinates*. In the symmetric power $S^2(V^*)$ we have

$$x^{i}x^{j} = x^{j}x^{i}$$
$$x^{i}\theta^{k} = \theta^{k}x^{i}$$
$$\theta^{k}\theta^{l} = -\theta^{l}\theta^{k}$$

that is, the x^i are commuting variables and the θ^k are anticommuting (or Grassmann) variables. With these notations,

$$\mathcal{F}(V) = \mathbb{C}[[x^1, \dots, x^{m_0}; \theta^1, \dots \theta^{m_1}]],$$

where $\theta^k \theta^l = -\theta^l \theta^k$.

5 Lie algebras

By definition, a Lie algebra is a vector space \mathfrak{g} endowed with a bracket

$$[\,,\,]\colon\mathfrak{g}\wedge\mathfrak{g}\to\mathfrak{g}$$

which satisfies the Jacobi identity. We can look at the Lie bracket as a map

$$[\,,\,]\colon S^2(\Pi\mathfrak{g})\to\Pi\mathfrak{g}.$$

Let $q := [,]^* : \Pi \mathfrak{g}^* \to S^2(\Pi \mathfrak{g}^*)$ be the dual map. The space $S^2(\Pi \mathfrak{g}^*)$ embeds into $\mathcal{F}(\Pi \mathfrak{g})$, so we can think of q as of a map

$$q \colon \Pi \mathfrak{g}^* \to \mathcal{F}(\Pi \mathfrak{g})$$

By forcing the Leibniz rule $\delta(\varphi_1\varphi_2) = \delta(\varphi_1)\varphi_2 + (-1)^{\overline{\varphi_1}}\varphi_1\delta(\varphi_2)$, we can extend q to a degree 1 derivative

$$\delta \colon \mathcal{F}(\Pi \mathfrak{g}) \to \mathcal{F}(\Pi \mathfrak{g}).$$

The operator δ is a differential, i.e., $\delta^2 = 0$. To see this, we only need to show that $\delta^2(\varphi_1 \cdots \varphi_n) = 0$ for any $\varphi_1, \ldots, \varphi_n \in \Pi \mathfrak{g}^*$. Since δ is a degree 1 derivative,

$$\delta(\varphi_1 \cdots \varphi_n) = (\delta \varphi_1) \varphi_2 \cdots \varphi_n - \varphi_1(\delta \varphi_2) \cdots \varphi_n + \cdots + (-1)^{n-1} \varphi_1 \varphi_2 \cdots (\delta \varphi_n)$$

Therefore

$$\delta^{2}(\varphi_{1}\cdots\varphi_{n}) = (\delta^{2}\varphi_{1})\varphi_{2}\cdots\varphi_{n} - (\delta\varphi_{1})(\delta\varphi_{2})\cdots\varphi_{n}$$

$$+\cdots + (-1)^{n-1}(\delta\varphi_{1})\varphi_{2}\cdots(\delta\varphi_{n})$$

$$+(\delta\varphi_{1})(\delta\varphi_{2})\cdots\varphi_{n} + \varphi_{1}(\delta^{2}\varphi_{2})\cdots\varphi_{n}$$

$$+\cdots + \varphi_{1}\varphi_{2}\cdots(\delta\varphi_{n})$$

$$= (\delta^{2}\varphi_{1})\varphi_{2}\cdots\varphi_{n} + \varphi_{1}(\delta^{2}\varphi_{2})\cdots\varphi_{n}$$

$$+\cdots + \varphi_{1}\varphi_{2}\cdots(\delta^{2}\varphi_{n})$$

so, in order to prove $\delta^2 = 0$ we just need to prove $\delta^2 \varphi = 0$ for any $\varphi \in \Pi \mathfrak{g}^*$. By definition, $\delta|_{\Pi \mathfrak{g}^*}$ is the dual of the Lie bracket, i.e.,

$$\langle \delta \varphi | g_1 \wedge g_2 \rangle = \langle \varphi | [g_1, g_2] \rangle.$$

One immediately computes

$$\langle \delta^2 \varphi | g_1 \wedge g_2 \wedge g_3 \rangle = \langle \varphi | [[g_1, g_2], g_3] + [g_2, g_3], g_1] + [[g_3, g_1], g_2] \rangle$$

So the $\delta^2 = 0$ is equivalent to the Jacobi identity.

We end this section by writing the coordinate expression of the differential δ . Let γ_i be a basis of $\mathfrak g$ and let c^i be the corresponding coordinates on $\Pi \mathfrak g$. Then

$$(\delta c^i)(\gamma_{j_0} \wedge \gamma_{k_0}) = \langle c^i | [\gamma_{j_0}, \gamma_{k_0}] \rangle = f^i_{j_0 k_0} = \frac{1}{2} \langle f^i_{jk} c^j c^k | \gamma_{j_0} \wedge \gamma_{k_0} \rangle,$$

where the f_{jk}^i are the structure constants of the Lie algebra \mathfrak{g} . Therefore

$$\delta c^i = \frac{1}{2} f^i_{jk} c^j c^k$$

that is,

$$\delta = \frac{1}{2} f^i_{jk} c^j c^k \frac{\partial}{\partial c^i}$$

6 A digression on L_{∞} -algebras

By the discussion in the above section, we can restate the definition of Lie algebra as follows: a Lie algebra is a vector space $\mathfrak g$ together with a degree 1 derivative δ on $\mathcal F(\Pi\mathfrak g)$ which is a differential. if one drops the hypothesis deg $\delta=1$, one obtains the definition of L_∞ -algebra. The structure of L_∞ -algebra can also be defined by means of multilinear operations. Indeed, if

$$\delta \colon \mathcal{F}(\Pi \mathfrak{g}) \to \mathcal{F}(\Pi \mathfrak{g})$$

is a derivation, then δ is completely determined by its restriction

$$\delta \colon \Pi \mathfrak{g}^* \to \mathcal{F}(\Pi \mathfrak{g})$$

Let δ_n be the projection of $\delta|_{\Pi\mathfrak{g}^*}$ on $S^n(\Pi\mathfrak{g}^*)$, and let

$$[\,,\ldots,\,]_n\colon S^n(\Pi\mathfrak{g})\to\Pi\mathfrak{g}$$

be the dual maps. Then the condition $\delta^2=0$ is equivalent to a family of quadratic relations among the brackets $[\,,\dots,\,]_n$, and an L_∞ -algebra can therefore be defined as a vector space $\mathfrak g$ endowed with a family of multilinear brackets

$$[\,,\ldots,\,]_n\colon \bigwedge^n \mathfrak{g} o \mathfrak{g}$$

with $\deg[\,,\ldots,\,]_n=n-1$ and satisfying certain quadratic relations. It is interesting to write down the first of these relations. With the above notations, $\delta^2=(\delta_1+\delta_2+\cdots)^2=\delta_1^2+(\delta_1\delta_2+\delta_2\delta_1)+\cdots$, so that $\delta^2=0$ implies

$$\delta_1^2 = 0$$

$$\delta_1 \delta_2 + \delta_2 \delta_1 = 0$$

$$\delta_1 \delta_3 + \delta_2^2 + \delta_3 \delta_1 = 0$$

and so on. Consider now the brackets

$$[\]_1 = \delta_1^* \colon \mathfrak{g} \to \mathfrak{g}$$
$$[\]_2 = \delta_2^* \colon \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$$
$$[\]_3 = \delta_3^* \colon \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$$

The above equations for the δ_i tell us that $[]_1$ is a differential on \mathfrak{g} . Moreover, if we set $d = []_1$, then

$$d[g_1, g_2]_2 = [dg_1, g_2]_2 + [g_1, dg_2]_2$$

i.e., d is a derivative with respect to the bracket $[]_2$. Finally,

$$\begin{split} [[g_1,g_2]_2,g_3]_2 + [[g_2,g_3]_2,g_1]_2 + [[g_3,g_1]_2,g_2]_2 = \\ d[g_1,g_2,g_3]_3 - [dg_1,g_2,g_3]_3 - [g_1,dg_2,g_3]_3 - [g_1,dg_2,g_3]_3 \end{split}$$

i.e., the Jacoby relation for the bracket $[]_2$ holds up to a homotopy given by the bracket $[]_3$. In this sense an L_∞ -algebra is a Lie algebra up to homotopy. Moreover, since the homotopies $[]_n$ with $n \geq 3$ are part of the data defining the structure of L_∞ -algebra, L_∞ -algebras are also called strong homotopy Lie algebras.

7 Lie algebras representations

Let V be a vector space and $\mathfrak g$ a Lie algebra. By definition, a representation of $\mathfrak g$ on V is a linear map

$$\rho \colon \mathfrak{g} \otimes V \to V$$

such that the induced map

$$\mathfrak{g} \to \operatorname{End}(V)$$

is a Lie algebra morphism. Equivalently, the diagram

$$(\mathfrak{g} \wedge \mathfrak{g}) \otimes V \xrightarrow{\iota} \mathfrak{g} \otimes (\mathfrak{g} \otimes V) \xrightarrow{\operatorname{id} \otimes \rho} \mathfrak{g} \otimes V$$

$$\downarrow \rho$$

$$\mathfrak{g} \otimes V \xrightarrow{\rho} V$$

is commutative, where the inclusion $\iota : \mathfrak{g} \wedge \mathfrak{g} \hookrightarrow \mathfrak{g} \otimes \mathfrak{g}$ is given by $g_1 \wedge g_2 \mapsto g_1 \otimes g_2 - g_2 \otimes g_1$. The three maps

$$0: S^{2}(V) \to 0$$

$$\rho: \mathfrak{g} \otimes V \to V$$

$$[,]: \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$$

can be seen as a map

$$\rho + [\,,\,] \colon S^2(V \oplus \Pi \mathfrak{g}) \to V \oplus \Pi \mathfrak{g}$$

by the isomorphism

$$S^2(V \oplus \Pi \mathfrak{g}) \simeq S^2(V) \oplus (\Pi \mathfrak{g} \otimes V) \oplus S^2(\Pi \mathfrak{g})$$

Denote by δ the map dual to $\rho + [,]$,

$$\delta \colon (V \oplus \Pi \mathfrak{g})^* \to S^2 ((V \oplus \Pi \mathfrak{g})^*)$$

and extend it to a degree 1 derivation

$$\delta \colon \mathcal{F}(V \oplus \Pi \mathfrak{g}) \to \mathcal{F}(V \oplus \Pi \mathfrak{g})$$

by forcing the Leibniz rule. Reasoning as for Lie algebras, one sees that δ is a differential, i.e., $\delta^2 = 0$. Indeed, for any $\varphi \in (V \oplus \Pi \mathfrak{g})^*$,

$$\langle \delta^2 \varphi | v_1 v_2 v_3 \rangle = \langle \varphi | 0 \rangle$$

$$\langle \delta^2 \varphi | g \otimes v_1 v_2 \rangle = \langle \varphi | 0 \rangle$$

$$\langle \delta^2 \varphi | (g_1 \wedge g_2) \otimes v \rangle = \langle \varphi | [g_1, g_2] \cdot v - g_1 \cdot (g_2 \cdot v) + g_2 \cdot (g_1 \cdot v) \rangle$$

$$\langle \delta^2 \varphi | g_1 \wedge g_2 \wedge g_3 \rangle = \langle \varphi | [[g_1, g_2], g_3] + [g_2, g_3], g_1] + [[g_3, g_1], g_2] \rangle$$

where we have written $g \cdot v$ for $\rho(g \otimes v)$. These equations show that the condition $\delta^2 = 0$ is equivalent to the Jacobi identity for \mathfrak{g} and to the fact that ρ is a representation. That is, a Lie algebra representation $(\mathfrak{g}, V, \rho, [\,,\,])$ can be seen as a superspace $V \oplus \Pi \mathfrak{g}$ endowed with a degree one derivative $\delta \colon \mathcal{F}(V \oplus \Pi \mathfrak{g}) \to \mathcal{F}(V \oplus \Pi \mathfrak{g})$ which is a differential, i.e., $\delta^2 = 0$.

The operator δ is called the *BRST operator*, and the cohomology of δ is called the *BRST cohomology*. Here BRST is the acronym for Becchi-Rouet-Stora-Tyutin.

A more refined analysis of BRST cohomology can be obtained taking care of the degrees. To begin with, recall that

$$\mathcal{F}(V \oplus \Pi \mathfrak{g}) = \bigoplus_{p,q} S^p(V^*) \otimes S^q(\Pi \mathfrak{g}^*) \simeq \bigoplus_{p,q} S^p(V^*) \otimes \bigwedge^q(\mathfrak{g}^*)$$

so that δ can be seen as an operator

$$\delta^{p,q} \colon S^p(V^*) \otimes \bigwedge^q \mathfrak{g}^* \to S^p(V^*) \otimes \bigwedge^{q+1} \mathfrak{g}^*$$

In particular, if we take p = 0 we obtain

$$\delta^{0,q} \colon \bigwedge^q \mathfrak{g}^* \to \bigwedge^{q+1} \mathfrak{g}^*$$

Dualizing this differential we obtain a differential

$$d_{q+1} \colon \bigwedge^{q+1} \mathfrak{g} \to \bigwedge^q \mathfrak{g}$$

which is easily seen to be the Chevalley-Eilenberg differential defining the Lie algebra cohomology of \mathfrak{g} . Another classical example is obtained by taking p=1. In this case one has

$$\delta^{1,q} \colon V^* \otimes \bigwedge^q \mathfrak{g}^* \to V^* \otimes \bigwedge^{q+1} \mathfrak{g}^*$$

Dualizing this differential we obtain a differential

$$d_{q+1} \colon V \otimes \bigwedge^{q+1} \mathfrak{g} \to V \otimes \bigwedge^{q} \mathfrak{g}$$

which is the differential defining the Lie algebra cohomology of $\mathfrak g$ with coefficients in the representation ρ .

As we did for Lie algebras, we end this section by writing the coordinate expression of the differential δ . Let e_i and γ_j be basis of V and \mathfrak{g} respectively and let v^i and c^i be the corresponding coordinates on V and $\Pi \mathfrak{g}$. Then

$$(\delta v^{i})(e_{j_{0}} \cdot e_{k_{0}} + e_{j_{1}} \otimes \gamma_{k_{1}} + \gamma_{j_{2}} \wedge \gamma_{k_{2}}) = \langle v^{i} | \rho(\gamma_{k_{1}}) e_{j_{1}} + [\gamma_{j_{2}}, \gamma_{k_{2}}] \rangle$$
$$= \rho^{i}_{j_{1} k_{1}} = \langle \rho^{i}_{j_{k}} v^{j} c^{k} | e_{j_{0}} \cdot e_{k_{0}} + e_{j_{1}} \otimes \gamma_{k_{1}} + \gamma_{j_{2}} \wedge \gamma_{k_{2}} \rangle,$$

where the ρ_{jk}^i are the structure constants of the Lie algebra representation $\rho \colon \mathfrak{g} \to \operatorname{End}(V)$. The computation we already did for Lie algebras give

$$(\delta c^{i})(e_{j_{0}} \cdot e_{k_{0}} + e_{j_{1}} \otimes \gamma_{k_{1}} + \gamma_{j_{2}} \wedge \gamma_{k_{2}}) = \langle c^{i} | \rho(\gamma_{k_{1}}) e_{j_{1}} + [\gamma_{j_{2}}, \gamma_{k_{2}}] \rangle$$

= $f^{i}_{j_{2}k_{2}} = \frac{1}{2} \langle f^{i}_{jk} c^{j} c^{k} | e_{j_{0}} \cdot e_{k_{0}} + e_{j_{1}} \otimes \gamma_{k_{1}} + \gamma_{j_{2}} \wedge \gamma_{k_{2}} \rangle,$

where the f_{jk}^i are the structure constants of the Lie algebra \mathfrak{g} . Therefore

$$\delta v^i = \rho^i_{jk} v^j c^k; \qquad \delta c^i = \frac{1}{2} f^i_{jk} c^j c^k$$

that is,

$$\delta = \rho^i_{jk} v^j c^k \frac{\partial}{\partial v^i} + \frac{1}{2} f^i_{jk} c^j c^k \frac{\partial}{\partial c^i}$$

8 Batalin-Vilkovisky algebras

Let now W be any superspace. The superspace $W \oplus \Pi W^*$ is naturally endowed with an odd non-degenerate pairing. Let Δ be the Laplace operator associated to this pairing. If x^i are coordinates on W (the *fields*) and x_i^+ are the corresponding coordinates on ΠW^* (the *antifields*), then

$$\Delta = \frac{\partial}{\partial x_i^+} \frac{\partial}{\partial x^i}$$

The operator Δ is called the Batalin-Vilkovisky Laplacian; note that, if Φ is a homogeneous function in $\mathcal{F}(W \oplus \Pi W^*)$, then $\Delta \Phi$ is also homogeneous and $\overline{\Delta \Phi} = \overline{\Phi} + 1 \mod 2$. It is immediate to compute that

$$\Lambda^2 = 0$$

Indeed,

$$\begin{split} \Delta^2 &= \frac{\partial}{\partial x_i^+} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x_j^+} \frac{\partial}{\partial x^j} \\ &= (-1)^{\overline{x^i} \cdot \overline{x_j^+} + \overline{x^i} \cdot \overline{x^j} + \overline{x_i^+} \cdot \overline{x_j^+} + \overline{x_i^+} \cdot \overline{x^j}} \frac{\partial}{\partial x_j^+} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x_i^+} \frac{\partial}{\partial x^i} \\ &= (-1)^{(\overline{x^i} + \overline{x_i^+})(\overline{x^j} + \overline{x_j^+})} \frac{\partial}{\partial x_j^+} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x_i^+} \frac{\partial}{\partial x^i}. \end{split}$$

Since the variables x^i and x_i^+ have opposite parity, $\overline{x^i} + \overline{x_i^+} = 1 \mod 2$, for any i. Therefore,

$$\Delta^2 = -\Delta^2$$

i.e., $\Delta^2 = 0$. The cohomology of $\mathcal{F}(W \oplus \Pi W^*)$ with respect to the BV-Laplacian is called BV-cohomology or Δ -cohomology.

Let now Φ and Ψ be two homogeneous functions on $W \oplus \Pi W^*$. Then

$$\begin{split} \Delta(\Phi \cdot \Psi) &= \frac{\partial}{\partial x_i^+} \frac{\partial}{\partial x^i} (\Phi \cdot \Psi) \\ &= \frac{\partial}{\partial x_i^+} \left(\frac{\partial \Phi}{\partial x^i} \cdot \Psi + (-1)^{\overline{x^i} \cdot \overline{\Phi}} \Phi \frac{\partial \Psi}{\partial x^i} \right) \\ &= \frac{\partial}{\partial x_i^+} \frac{\partial \Phi}{\partial x^i} \cdot \Psi + (-1)^{(\overline{x^i} + \overline{\Phi}) \overline{x_i^+}} \frac{\partial \Phi}{\partial x^i} \frac{\partial \Psi}{\partial x_i^+} + (-1)^{\overline{x^i} \cdot \overline{\Phi}} \frac{\partial \Phi}{\partial x_i^+} \frac{\partial \Psi}{\partial x^i} + \\ &\quad + (-1)^{(\overline{x^i} + \overline{x_i^+}) \overline{\Phi}} \Phi \frac{\partial}{\partial x_i^+} \frac{\partial \Psi}{\partial x^i} \\ &= (\Delta \Phi) \cdot \Psi + (-1)^{\overline{\Phi}} \{\Phi, \Psi\} + (-1)^{\overline{\Phi}} \Phi \cdot \Delta \Psi \end{split}$$

where $\{\Phi, \Psi\}$ is the so-called BV-bracket, defined by

$$\{\Phi,\Psi\} = (-1)^{\overline{x_i^+}\cdot\overline{\Phi}} \frac{\partial\Phi}{\partial x_i^+} \frac{\partial\Psi}{\partial x^i} - (-1)^{(\overline{\Phi}+1)(\overline{\Psi}+1) + \overline{x_i^+}\cdot\overline{\Psi}} \frac{\partial\Psi}{\partial x_i^+} \frac{\partial\Phi}{\partial x^i} \frac{\partial\Phi}{\partial x^i} \frac{\partial\Phi}{\partial x_i^+} \frac{\partial\Phi}{\partial x_i^+}$$

The BV-bracket is best expressed using left and right derivatives: for a homogeneous vector v in $W \oplus \Pi W^*$, set

$$\overrightarrow{\partial}_v \Phi = \partial_v \Phi$$

$$\overleftarrow{\partial}_v \Phi = (-1)^{\overline{v} \cdot \overline{\Phi}} \partial_v \Phi$$

With these notations, the BV-bracket reads

$$\{\Phi,\Psi\} = \frac{\overleftarrow{\partial} \Phi}{\partial x_{+}^{i}} \frac{\overrightarrow{\partial} \Psi}{\partial x_{-}^{i}} - (-1)^{(\overline{\Phi}+1)(\overline{\Psi}+1)} \frac{\overleftarrow{\partial} \Psi}{\partial x_{+}^{i}} \frac{\overrightarrow{\partial} \Phi}{\partial x_{-}^{i}}$$

The Leibniz rule for derivatives gives

$$\begin{split} \{\Phi,\Psi\cdot\Upsilon\} &= (-1)^{\overline{x_i^+}\cdot\overline{\Phi}} \frac{\partial\Phi}{\partial x_i^+} \frac{\partial\Psi}{\partial x^i} \Upsilon + (-1)^{\overline{x_i^+}\cdot\overline{\Phi}+(\overline{\Phi}+1)\overline{\Psi}} \Psi \frac{\partial\Phi}{\partial x_i^+} \frac{\partial\Upsilon}{\partial x^i} + \\ &- (-1)^{(\overline{\Phi}+1)(\overline{\Psi}+\overline{\Upsilon}+1)+\overline{x_i^+}(\overline{\Psi}+\overline{\Upsilon})} \frac{\partial\Psi}{\partial x_i^+} \Upsilon \frac{\partial\Phi}{\partial x^i} + \\ &- (-1)^{(\overline{\Phi}+1)(\overline{\Psi}+\overline{\Upsilon}+1)+\overline{x_i^+}(\overline{\Psi}+\overline{\Upsilon})+\overline{x_i^+}\cdot\overline{\Upsilon}} \Psi \frac{\partial\Upsilon}{\partial x_i^+} \frac{\partial\Phi}{\partial x^i} \\ &= \{\Phi,\Psi\}\Upsilon + (-1)^{(\overline{\Phi}+1)\overline{\Psi}} \Psi \{\Phi,\Upsilon\} \end{split}$$

i.e., the BV-bracket satisfies an odd Poisson identity: for any homogeneous $\Phi \in \mathcal{F}(W \oplus \Pi W^*)$, the operator $\mathrm{ad}_{\Phi} = \{\Phi, -\}$ is a derivative of degree $\overline{\Phi} + 1$.

The data $\mathcal{F}(W \oplus \Pi W^*)$, \cdot , Δ and $\{,\}$ are the basic example of BV-algebra. More generally, a BV-algebra is $(\mathcal{A}, \cdot, \Delta, \{,\})$, where

1. A is a superspace;

- 2. $\cdot: A \otimes A \to A$ is an associative and graded-commutative multiplication:
- 3. $\Delta : \mathcal{A} \to \mathcal{A}$ is an odd differential (here odd means that Δ changes the parity of a homogeneous element);
- 4. $\{,\}: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a bilinear operation such that

$$\Delta(\Phi \cdot \Psi) = (\Delta\Phi) \cdot \Psi + (-1)^{\overline{\Phi}} \{\Phi, \Psi\} + (-1)^{\overline{\Phi}} \Phi \cdot \Delta\Psi$$

and

$$\{\Phi, \Psi \cdot \Upsilon\} = \{\Phi, \Psi\}\Upsilon + (-1)^{(\overline{\Phi}+1)\overline{\Psi}}\Psi\{\Phi, \Upsilon\}$$

The operations \cdot , Δ and $\{\ ,\ \}$ are called, respectively, multiplication, BV-Laplacian and BV-bracket. Note that the parity of $\{\Phi,\Psi\}$ is $\overline{\Phi} + \overline{\Psi} + 1$ mod 2.

We will now derive some consequences of the compatibility between the three operations in a BV-algebra. To begin with, the BV-bracket is anticommutative, up to a parity change:

$$\{\Psi,\Phi\} = -(-1)^{(\overline{\Psi}+1)(\overline{\Phi}+1)}\{\Phi,\Psi\}$$

Indeed, $\Psi \cdot \Phi = (-1)^{\overline{\Psi} \cdot \overline{\Phi}} \Phi \cdot \Psi$, so that

$$\begin{split} \Delta(\Psi \cdot \Phi) &= (-1)^{\overline{\Psi} \cdot \overline{\Phi}} \Delta(\Phi \cdot \Psi) \\ &= (-1)^{\overline{\Psi} \cdot \overline{\Phi}} (\Delta\Phi) \cdot \Psi + (-1)^{\overline{\Phi}(\overline{\Psi} + 1)} \{\Phi, \Psi\} + (-1)^{\overline{\Phi}(\overline{\Psi} + 1)} \Phi \cdot \Delta\Psi \end{split}$$

Since the BV-Laplacian changes the parity of a homogeneous function, we can rewrite this as

$$\Delta(\Psi\cdot\Phi)=(-1)^{\overline{\Psi}}\Psi\cdot\Delta\Phi+(-1)^{\overline{\Phi}(\overline{\Psi}+1)}\{\Phi,\Psi\}+(\Delta\Psi)\cdot\Phi$$

On the other hand,

$$\Delta(\Psi \cdot \Phi) = (\Delta \Psi) \cdot \Phi + (-1)^{\overline{\Psi}} \{ \Psi, \Phi \} + (-1)^{\overline{\Psi}} \Psi \cdot \Delta \Phi$$

so that, equating the right hand sides of the two equations above, we find

$$(-1)^{\overline{\Phi}(\overline{\Psi}+1)}\{\Phi,\Psi\} = (-1)^{\overline{\Psi}}\{\Psi,\Phi\}$$

which is the equation we were looking for. Note that odd anticommutativity for the BV-bracket on $\mathcal{F}(W \oplus \Pi W^*)$ could be immediately checked from the definition of the bracket.

The compatibility equation between the three operations in a BV-algebra shows that the Laplacian is not a derivative with respect to the multiplication: indeed, the BV-bracket measures precisely how much Δ fails to be a derivative. Yet, the Laplacian is a derivative with respect to the bracket:

$$\Delta\{\Phi,\Psi\} = \{\Delta\Phi,\Psi\} + (-1)^{\overline{\Phi}+1}\{\Phi,\Delta\Psi\}$$

This is an immediate consequence of the trivial identity $\Delta^2(\Phi \cdot \Psi) = 0$. Finally, applying the BV-Laplacian to the odd Poisson identity $\{\Phi, \Psi \cdot \}$ $\Upsilon\}=\{\Phi,\Psi\}\Upsilon+(-1)^{(\overline{\Phi}+1)\overline{\Psi}}\Psi\{\Phi,\Upsilon\}$ one finds the odd Jacobi identity for the BV-bracket:

$$\{\Phi,\{\Psi,\Upsilon\}\}=\{\{\Phi,\Psi\},\Upsilon\}+(-1)^{(\overline{\Phi}+1)(\overline{\Psi}+1)}\{\Psi,\{\Phi,\Upsilon\}\}$$

As a concluding remark, note that by the compatibility equation between the BV-Laplacian, the multiplication and the BV-bracket

$$\Delta(\Phi \cdot \Psi) = (\Delta\Phi) \cdot \Psi + (-1)^{\overline{\Phi}} \{\Phi, \Psi\} + (-1)^{\overline{\Phi}} \Phi \cdot \Delta\Psi$$

one can express the BV-bracket entirely in terms of the of the multiplication and of the BV-Laplacian. The odd Poisson identity is then translated into the following *seven-terms relation*:

$$\begin{split} \Delta(\Phi\Psi\Upsilon) + (\Delta\Phi)\Psi\Upsilon + (-1)^{\overline{\Phi}}\Phi(\Delta\Psi)\Upsilon + (-1)^{\overline{\Phi}+\overline{\Psi}}\Phi\Psi(\Delta\Upsilon) = \\ \Delta(\Phi\Psi)\Upsilon + (-1)^{\overline{\Phi}}\Phi\Delta(\Psi\Upsilon) + (-1)^{(\overline{\Phi}+1)\overline{\Psi}}\Psi\Delta(\Phi\Upsilon) \end{split}$$

Therefore, one could define BV algebras as the data $(A, \cdot, \Delta,)$, where A is a superspace, $\cdot : A \otimes A \to A$ is an associative and graded-commutative multiplication, and $\Delta : A \to A$ is an odd differential such that the seventerms relation hold. The BV-bracket would then be defined by the formula

$$\{\Phi,\Psi\} = (-1)^{\overline{\Phi}} \Delta(\Phi \cdot \Psi) + (-1)^{\overline{\Phi}+1} (\Delta\Phi) \cdot \Psi - \Phi \cdot \Delta\Psi$$

On the other hand, the BV-bracket is such an important operation in the Batalin-Vilkovisky formalism, that we preferred to make it part of the definition of BV-algebra.

9 BV cohomology, Lagrangian submanifolds and the quantum master equation

This is the most geometrical part of this short note. Being just an introduction, we will be very sketchy and invite the reader to look at the details into [Schw] and [AKSZ]. The starting point is that, for any superspace W, the superspace $W \oplus \Pi W^*$ has a canonical odd symplectic structure. This is very familiar from classic differential geometry: for any vector space V, the vector space $V \oplus V^*$ has a canonical symplectic structure. Since we are dealing with an (odd) symplectic space, it is meaningful to consider Lagrangian submanifolds of $W \oplus \Pi W^*$: these are just isotropic (super-)submanifolds of maximal dimension. The volume form $dx^1 dx^2 \cdots dx^n dx_1^+ dx_2^+ \ldots dx_n^+$ induces well defined volume forms on the Lagrangian submanifolds of $W \oplus \Pi W^*$, so the functionals

$$\int_{\mathcal{L}}$$

on $\mathcal{F}(W \oplus \Pi W^*)$ are defined for any Lagrangian $\mathcal{L} \subseteq W \oplus \Pi W^*$. Here comes the main theorem of the BV-formalism. Let $\Phi \in \mathcal{F}(W \oplus \Pi W^*)$.

If
$$\Delta\Phi=0$$
, then $\int_{\mathcal{L}}\Phi$ depends only on the homology class of \mathcal{L} .
Moreover, if $\Phi=\Delta\Psi$, then $\int_{\mathcal{L}}\Phi=0$ for any Lagrangian \mathcal{L} .

This can be thought as a BV version of the familiar Stokes' theorem. In particular, we have a pairing between homology classes of Lagrangian submanifolds and Δ -cohomology classes of functions on the superspace $W \oplus \Pi W^*$.

As in classical symplectic manifolds, for any smooth function F on W, the sub-manifold \mathcal{L}_F defined by the equations

$$x_i^+ = \frac{\partial F}{\partial x^i}$$

is Lagrangian. Moreover, for any two functions F_0 and F_1 the homotopy $F_t = tF_1 + (1-t)F_0$ shows that \mathcal{L}_{F_0} and \mathcal{L}_{F_1} are in the same homology class. Therefore

$$\int_{\mathcal{L}_{F_0}} \Phi = \int_{\mathcal{L}_{F_1}} \Phi$$

for any Δ -closed Φ . In particular, the subspace W of $W \oplus \Pi W^*$ is defined by the equations

$$x_i^+ = 0$$

so it is the Lagrangian subspace defined by the function $F \equiv 0$. It follows that, for any Δ -closed function Φ and any smooth function F,

$$\int_W \Phi = \int_{\mathcal{L}_F} \Phi.$$

The function F is called the gauge fixing fermion.

Integrands we are interested in are usually of the form

$$\Phi = \Psi e^{\frac{i}{\hbar}S}$$

We begin by assuming that $\Psi = 1$, so that the equation $\Delta \Phi = 0$ becomes

$$0 = \Delta e^{\frac{i}{\hbar}S} = \Delta \left(\sum_{n=0}^{\infty} \frac{(iS)^n}{\hbar^n n!} \right) = \left(\frac{i}{\hbar} \Delta S - \frac{1}{2\hbar^2} \{S, S\} \right) e^{\frac{i}{\hbar}S}$$

so the condition $\Delta e^{\frac{i}{\hbar}S} = 0$ is equivalent to the quantum master equation

$$\{S,S\} - 2i\hbar\Delta S = 0$$

Note that, since S is even, the bracket $\{S,S\}$ is non trivial. If S can be expanded as a series in \hbar ,

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$$

then the quantum master equation becomes the sequence of equations

$$\{S_0, S_0\} = 0$$
$$\{S_0, S_1\} = i\Delta S_0$$
$$\{S_0, S_2\} + \frac{1}{2}\{S_1, S_1\} = i\Delta S_1$$

٠.

The first equation in the above list is called the *master equation*; since S_0 is even, it is a non-trivial equation. We will see in the next section how to relate solutions of the master equation with Lie algebra representations. But now, let us go back to the problem of calculating the Gaussian integral

$$\langle\!\langle \Psi \rangle\!\rangle = \int_W \Psi \, e^{\frac{i}{\hbar}S}$$

via the BV formalism. Assume that S is a solution of the quantum master equation. In order to apply the BV formalism, we want

$$\Delta\left(\Psi \, e^{\frac{i}{\hbar}S}\right) = 0$$

which is equivalent to

$$\Delta\Psi + \frac{i}{\hbar}\{S, \Psi\} = 0$$

Let Ω be the operator

$$\Omega = -i\hbar\Delta + \mathrm{ad}_S$$

then the BV formalism applies to all the functions Ψ in ker Ω . The operator Ω is actually a differential; indeed,

$$\begin{split} \Omega^2 \Psi &= \Omega \left(-i\hbar \Delta \Psi + \{ S, \Psi \} \right) \\ &= -\hbar^2 \Delta^2 \Psi - i\hbar \Delta \{ S, \Psi \} - i\hbar \{ S, \Delta \Psi \} + \{ S, \{ S, \Psi \} \} \\ &= -i\hbar \{ \Delta S, \Psi \} + \frac{1}{2} \{ \{ S, S \}, \Psi \} \\ &= \frac{1}{2} \{ \{ S, S \} - 2i\hbar \Delta S, \Psi \} = 0. \end{split}$$

Moreover, if Ψ is Ω -exact, then

$$\langle\!\langle \Psi \rangle\!\rangle = \int_W (\Omega \Psi_0) \, e^{\frac{i}{\hbar}S} = -i\hbar \int_W \Delta \left(\Psi_0 \, e^{\frac{i}{\hbar}S} \right) = 0$$

Therefore, the expectation value is actually a linear functional on the Ω -cohomology of $\mathcal{F}(W \oplus \Pi W^*)$; the Ω -cohomology classes are called *observables* of the theory. The operator Ω and its cohomology are called, respectively, the quantum BRST operator and the quantum BRST cohomology.

10 From Lie algebras representations to solutions of the master equation

Recall that to a Lie algebra representation $\rho \colon \mathfrak{g} \otimes V \to V$ is associated a BRST differential δ on $\mathcal{F}(V \oplus \Pi \mathfrak{g})$. Set

$$W=V\oplus\Pi\mathfrak{g}$$

and consider the superspace

$$W \oplus \Pi W^*$$

As we have seen, $\mathcal{F}(W \oplus \Pi W^*)$ has a natural structure of BV-algebra. Let S_0 be any element in $\mathcal{F}(W)$. It can be considered as an element of $\mathcal{F}(W \oplus \Pi W^*)$ by the projection $W \oplus \Pi W^* \to W$. Clearly, as an element of $\mathcal{F}(W \oplus \Pi W^*)$, the function S_0 satisfies the master equation: $\{S_0, S_0\} = 0$; indeed, S_0 is independent of the coordinates of vectors in ΠW^* . Therefore, the operator $\mathrm{ad}_{S_0} = \{S_0, -\}$ is a differential on $\mathcal{F}(W \oplus \Pi W^*)$ which induces the zero differential on $\mathcal{F}(W)$. Moreover, if \mathfrak{g} is a Lie algebra of infinitesimal symmetries of the action S_0 , then $\delta S_0 = 0$.

Let now S_1 be the quadratic function on $W \oplus \Pi W^*$ defined by

$$S_1(w+\omega) = \langle \delta w | \omega \rangle$$

If x^1,\dots,x^n are coordinates on W and x_i^+,\dots,x_n^+ are the dual coordinates on $\Pi W^*,$ then

$$S_1 = x_i^+ \delta x^i$$

Clearly,

$$\{S_1, \Phi\} = \delta\Phi$$

for any $\Phi \in \mathcal{F}(W)$. Moreover,

$$\delta S_1 = (-1)^{\overline{x}_i^+} x_i^+ \delta^2 x^i = 0,$$

since $\delta^2 = 0$. We compute

$$\{S_1, S_1\} = \{S_1, x_i^+ \delta x^i\}$$

$$= \{S_1, x_i^+ \} \delta x^i + (-1)^{\overline{x}_i^+} x_i^+ \{S_1, \delta x^j\}$$

$$= \frac{\partial S_1}{\partial x^i} \delta x^1 + (-1)^{\overline{x}_i^+} x_i^+ \delta^2 x^i$$

$$= 2\delta S_1 = 0$$

Since $\{S_1, S_0\} = \delta S_0$, if \mathfrak{g} is a Lie algebra of infinitesimal symmetries of the action S_0 , then $\{S_1, S_0\} = 0$. So, if we set

$$S = S_0 + \hbar S_1$$

then

$$\{S, S\} = 0$$

that is S is a solution of the master equation. As a consequence, $\mathrm{ad}_S = \{S, -\}$ is a differential on $\mathcal{F}(W \oplus \Pi W^*)$. Moreover $\{S, -\}$ induces the differential $\hbar\delta$ on $\mathcal{F}(W)$, since

$$\{S, \Phi\} = \{S_0, \Phi\} + \hbar\{S_1, \Phi\} = \hbar\{S_1, \Phi\} = \hbar\delta\Phi$$

for any $\Phi \in \mathcal{F}(W)$.

11 On-shell and off-shell representations of Lie algebras

In the previous section we have seen how to write a solution to the master equation starting from a Lie algebra representation and a function S_0

for which the Lie algebra \mathfrak{g} was an algebra of infinitesimal symmetries. Moreover, the solution $S=S_0+\hbar S_1$ was first order in the antifields. We will now show how to go the other way round, i.e., how to associate a Lie algebra representation and a function S_0 for which the Lie algebra is an algebra of infinitesimal symmetries to a solution to the master equation which is first order in the antifields.

Let $S = S_0 + \hbar S_1$ be a solution of the master equation. We stress the fact that S_k is of degree k in the antifields, for k = 0, 1 by writing

$$S = S_0(x^1, \dots, x^n) + \hbar x_i^+ Q^i(x^1, \dots, x^n)$$

Since S_0 does not depend on the antifields, the master equation $\{S,S\}$ is reduced to the two equations

$${S_1, S_0} = 0$$

 ${S_1, S_1} = 0$

The second equation of this pair tells that ad_{S_1} is a differential on $\mathcal{F}(W \oplus \Pi W^*)$. Moreover, since S_1 is first-order in the antifields, ad_{S_1} maps $\mathcal{F}(W)$ to itself. Let $\delta \colon \mathcal{F}(W) \to \mathcal{F}(W)$ be the differential defined by the restriction of ad_{S_1} to $\mathcal{F}(W)$. It is immediate to compute

$$\delta x^{i} = \{S_{1}, x^{i}\} = Q^{i}(x^{1}, \dots, x^{n})$$

so that, if $\deg Q_i=2$, then δ is a degree one derivative on $\mathcal{F}(W)$ which is a differential. As we have shown in section 7, this defines a Lie algebra representation $\mathfrak{g} \to \operatorname{End}(V)$, where V and \mathfrak{g} are defined by $W=V\oplus \Pi \mathfrak{g}$, i.e., $V=W_0$ and $\mathfrak{g}=\Pi W_1$. Note that the equations $\delta x^i=Q^i(x^1,\ldots,x^n)$ imply that $S_1=x_i^+\delta x^i$, i.e., S_1 has the form seen in section 10.

Finally, the equation $\{S_1, S_0\}$ gives $\delta S_0 = 0$, that is, the Lie algebra \mathfrak{g} is an algebra of infinitesimal symmetries for S_0 . If we drop the condition $\deg Q_i = 2$ then we get into the realm of homotopy Lie algebras representations, see section 6 above.

If we drop the hypothesis S being of degree 1 in the antifields, and write

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$$

with S_k of degree k in the antifields, then the master equation $\{S, S\} = 0$ becomes the sequence of equations

$${S_0, S_1} = 0$$

 ${S_1, S_1} = -2{S_0, S_2}$
...

The second equation in the list above tells that ad_{S_1} is no more a differential. On the other hand, since S_0 does not depend on the antifields, at critical points of S_0 , i.e., at points p in W such that

$$\left. \frac{\partial S_0}{\partial x^i} \right|_p = 0, \qquad \forall i,$$

one has $\{S_0, S_2\} = 0$. This means that $\delta = \mathrm{ad}_{S_1}$ is a differential when restricted to the set of critical points of S_0 . One refers to this phenomenon

by saying that δ is differential on-shell. When δ is a differential on the whole of W, as in the case considered before, one says that δ is a differential off-shell.

12 Traceless representations and the quantum master equation

In the above section we have seen how to build a solution to the master equation from a function S_0 and a Lie algebra of infinitesimal symmetries for this function. We now show how, if the representation $\rho\colon \mathfrak{g}\to V$ inducing the infinitesimal symmetries of the action is traceless, then the function $S=S_0+\hbar S_1$ is actually a solution to the quantum master equation.

Recall that the quantum master equation is the equation

$$2i\hbar\Delta S - \{S, S\} = 0$$

If $S = S_0 + \hbar S_1$ as in the previous section, the quantum master equation is reduced to

$$\Delta S_1 = 0$$

It is immediate to compute

$$\Delta S_1 = \Delta(x_i^+ \delta x^i) = \frac{\partial}{\partial x_i} (\delta x^i) = \text{div}\delta$$

i.e., ΔS_1 is the divergence of the vector field δ ; since δ is a vector field on $V \oplus \Pi \mathfrak{g}$, its divergence is a function on this space. Now, recall that

$$\delta = \rho_{jk}^{i} v^{j} c^{k} \frac{\partial}{\partial v^{i}} + \frac{1}{2} f_{jk}^{i} c^{j} c^{k} \frac{\partial}{\partial c^{i}}$$

so that

$$\operatorname{div}\delta = (\rho_{ik}^i + f_{ik}^i)c^k,$$

i.e.,

$$\delta \colon (v, \mathfrak{g}) \mapsto \operatorname{Tr}(\operatorname{ad}_g) + \operatorname{Tr}(\rho_g)$$

Therefore, if the adjoint representation of $\mathfrak g$ on itself and the representation ρ are traceless, then $S=S_0+\hbar S_1$ is a solution to the quantum master equation. Since $S\big|_V=S_0\big|_V$, we have achieved our goal to extend $e^{\frac{i}{\hbar}S_0}$ to a Δ -closed function.

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